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Some Double Integral Representations For The Exton's Triple Hypergeometric Function X_9 Involving Gauss Hypergeometric Function, Generalized Kampé Dé Fériét Function And Exton's Function

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ABSTRACT
In recent years several authors have made significant contributions in the development of the work due to Exton. In the present paper double integral representations of Eulerian kind for the triple hypergeometric function $\mathbf{X_9}$ have been established which involve Gauss hypergeometric function ${}_2\mathbf{F}_1$, generalized, kampé dé Fériét function, Lauricella function and Exton's function $\mathbf{X_9}$ itself in their kernels.
Keywords - Exton's triple hypergeometric function, Gauss hypergeometric function, kampé dé Fériét function, Appell function, Laplace integral.

I. INTRODUCTION

In 1982 Exton [2] has introduced Laplace integral representations for each triple hypergeometric series X_1 to X_{20}. After then several authors [1, 4, 5, 6] made their contribution in the extension and development of theory of double integral and Exton's work. In this paper some new double integral representations of Eulerian kind for the Exton's function X_9 have been established which involve Gauss hypergeometric function, generalized, kampé dé Fériét function, Exton function and Appell function in their integrals.

Exton [2] defined the following series representation and Laplace integral representation for the Exton function X_9.

$$X_9(a,b;c;x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+n}\,(b)_{n+2p}\,x^m\,y^n\,z^p}{(c)_{m+n+p}\quad m!\;n!\;p!} \tag{1.1}$$

Srivastava and Karlsson [3] have given the precise three dimensional region of convergence of (1.1):

$$r<\frac{1}{4}<t<\frac{1}{4}\wedge s<\frac{1}{2}+\frac{1}{2}\sqrt{(1-4r)(1-4t)},$$
$$|x|<r,|y|<s,|z|<t \tag{1.2}$$

Where positive quantities r,s,t are associated radii of convergence.

$$X_9(a,b;c;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty e^{-s-t}\; s^{a-1}\; t^{b-1}$$
$$\times\;{}_0F_1(-;c;\;xs^2+yst+zt^2)\,dsdt \tag{1.3}$$

$\mathrm{Re}(a)>0, Re(b)>0$

II. RESULTS REQUIRED

The well known integral formulas [7] have been used in the present investigations.

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[1+ct+d(1-t)]^{-(\alpha+\beta)}\,dt$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)(1+c)^{\alpha}(1+d)^{\beta}} \tag{2.1}$$

$\mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0$ and the constants λ and μ are such that none of the expression $1+c, 1+d, [1+ct+d(1+t)]$, where $0\le t\le 1$, are zero.

$$\int\int u^{\alpha-1}v^{\beta-1}(1-u-v)^{\gamma-1}\,dudv$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)} \tag{2.2}$$

Where $u\ge 0, v\ge 0$ and $u+v\le 1, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$ and $\mathrm{Re}(\gamma)>0$.

$$\int_a^b (x-a)^{\alpha-1}\;(b-x)^{\beta-1}\,(x-c)^{-(\alpha+\beta)}\,dx$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)(b-c)^{-\alpha}\;(a-c)^{-\beta}\,(b-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \tag{2.3}$$

$a<c<b, Re(\alpha)>0, Re(\beta)>0$

$$B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\;dt, & (\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0)\\ \displaystyle\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\mathrm{Re}(\alpha)<0,\mathrm{Re}(\beta)<0;\;\alpha,\beta\in c\backslash z_0^-)\end{cases} \tag{2.4}$$

Binomial theorem

$$(1+x)^{-a}=\sum_{r=0}^{\infty}\frac{(a)_r\;(-x)^r}{r!} \tag{2.5}$$

$$(m-n)! = \frac{(-1)^n \ m!}{(-m)_n}, \qquad 0 \le n \le m \tag{2.6}$$

Lemma

$$\sum_{r=0}^{\infty}\sum_{p=0}^{\infty} A(p,r) = \sum_{r=0}^{\infty}\sum_{p=0}^{r} A(p,r-p) \tag{2.7}$$

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \tag{2.8}$$

$$(\alpha)_k = 2^{2k}\left(\frac{\alpha}{2}\right)_k\left(\frac{\alpha}{2}+\frac{1}{2}\right)_k \tag{2.9}$$

III. MAIN RESULTS

$$X_9(a,b;c;x,y,z) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\iint u^{a-1}v^{b-1}(1-u-v)^{c-1} \times {}_2F_1\left(\frac{a+b+c}{2},\frac{a+b+c+1}{2};c;4(u^2x+uvy+v^2z)\right)dudv \tag{3.1}$$

$u \ge 0, v \ge 0, u+v \le 1, \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0, Re(c) > 0$

here ${}_2F_1$ denotes the well known Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{z^r}{r!}$$

The above series is convergent for all values of z provided $|z| < 1$ and divergent if $|z| > 1$. When $z = 1$, the series is convergent if $\mathrm{Re}(c-a-b) > 0$ and divergent if $\mathrm{Re}(c-a-b) \le 0$. When $z = -1$, the series is absolutely convergent when $\mathrm{Re}(c-a-b>0$ and is convergent but not absolutely when $-1 < Re(c-a-b) \le 0$ and divergent when $\mathrm{Re}(c-a-b) < -1$.

$$X_9(a,b;c_1+c_2;x,y,z) = \frac{\Gamma(a+b)\Gamma(c_1+c_2)}{\Gamma(a)\Gamma(b)\Gamma(c_1)\Gamma(c_2)}\frac{(\beta-\gamma)^a(\alpha-\gamma)^b(\mu-\nu)^{c_1}(\lambda-\nu)^{c_2}}{(\beta-\alpha)^{a+b-1}(\mu-\lambda)^{c_1+c_2-1}} \times \int_{\lambda}^{\mu}\int_{\alpha}^{\beta}(x-\alpha)^{a-1}(\beta-x)^{b-1}(x-\gamma)^{-(a+b)}(y-\lambda)^{c_1-1} \times (\mu-y)^{c_2-1}(y-\nu)^{-(c_1+c_2)}$$
$$\times\left[F^{2:0:0}_{0:1:1}\left\{\begin{matrix}\frac{a+b}{2},\frac{a+b+1}{2}: -: - ; \\ -: c_1 : c_2 ;\end{matrix}\ \sigma_1\frac{(x-\alpha)^2(y-\lambda)x}{(x-\gamma)^2(y-\nu)}, \sigma_2\frac{(x-\alpha)(\beta-x)(\mu-y)y}{(x-\gamma)^2(y-\nu)} + \sigma_3\frac{(\beta-x)^2(\mu-y)z}{(x-\gamma)^2(y-\nu)}\right\}\right]dxdy \tag{3.2}$$

Here $\alpha < \gamma < \beta;\ \lambda < \nu < \mu; Re(a) > 0, Re(b) > 0; Re(c_1) > 0, Re(c_2) > 0$

$$\sigma_1 = \frac{4(\mu-\nu)(\beta-\gamma)^2}{(\mu-\lambda)(\beta-\alpha)^2},$$

$$\sigma_2 = \frac{4(\alpha-\gamma)(\beta-\gamma)(\lambda-\nu)}{(\mu-\lambda)(\beta-\alpha)^2},$$

$$\sigma_3 = \frac{4(\alpha-\gamma)^2(\lambda-\nu)}{(\mu-\lambda)(\beta-\alpha)^2}$$

Here $F^{2:0:0}_{0:1:1}$ denotes generalized kampé dé Fériét function (3, p.27, results 28, 29, sec. 1.3)

$$X_9(a,b;c;x,y,z) = \frac{(1+\lambda)^a(1+\lambda')^b(1+\mu)^{\alpha-a}(1+\mu')^{\beta-b}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(a)\Gamma(b)\Gamma(\alpha-a)\Gamma(\beta-b)} \times \int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{\alpha-a-1}(1-v)^{\beta-b-1} \times [1+\lambda u+\mu(1-u)]^{-\alpha}[1+\lambda' v+\mu'(1-v)]^{-\beta} \times [X_9(\alpha,\beta;c;\ \sigma_1 x,\sigma_2 y,\sigma_3 z)]dudv \tag{3.3}$$

$\mathrm{Re}(\alpha) > Re(a) > 0; \mathrm{Re}(\beta) > Re(b) > 0$ and the constant λ,μ and λ',μ' are such that none of the expression $1+\lambda, 1+\mu, 1+\lambda u+\mu(1-u)$ and $1+\lambda', 1+\mu', 1+\lambda' v+\mu'(1-v)$, where $0 \le u \le 1$ and $0 \le v \le 1$ respectively, are zero.

$$\sigma_1: \left[\frac{(1+\lambda)u}{1+\lambda u+\mu(1-u)}\right]^2,$$

$$\sigma_2: \left[\frac{(1+\lambda)(1+\lambda')uv}{\{1+\lambda u+\mu(1-u)\}\{1+\lambda' v+\mu'(1-v)\}}\right],$$

$$\sigma_3: \left[\frac{(1+\lambda')v}{1+\lambda' v+\mu'(1-v)}\right]^2$$

$$X_9(a,b;c;x,y,z) = \frac{\Gamma(\lambda)}{\Gamma(a)\Gamma(\lambda-a)}\frac{\Gamma(\mu)}{\Gamma(b)\Gamma(\mu-b)} \times \int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{\lambda-a-1}(1-v)^{\mu-b-1} \times X_9(\lambda,\mu;c;\ xu^2,yuv,zv^2)dudv \tag{3.4}$$

$\mathrm{Re}(\lambda) > Re(a) > 0; \mathrm{Re}(\mu) > Re(b) > 0$

Where X_9 denotes Exton function (1.1) and its condition of convergence is given in (1.2).

$$X_9(a,b;c_1+c_2;x,y,z) = \frac{\Gamma(a+b)\Gamma(c_1+c_2)}{\Gamma(a)\Gamma(b)\Gamma(c_1)\Gamma(c_2)} \times \int_0^1\int_0^1 \xi^{c_1-1}(1-\xi)^{c_2-1}u^{a-1}(1-u)^{b-1} \times F_4\left(\frac{a+b}{2},\frac{a+b+1}{2},c_1,c_2;4xu^2\xi,4(1-\xi) \times [yu(1-u)+z(1-u)^2]\right)dud\xi \tag{3.5}$$

$\mathrm{Re}(c_1) > 0, \mathrm{Re}(c_2) > 0, \mathrm{Re}(a) > 0, \mathrm{Re}(b) > 0$

Where F_4 denote Appell function [3, pp 22, ch.1, result(5)].

IV. PROOF OF THE RESULTS

In order to prove the result (3.1), begin with right hand side by expressing the series definition of Gauss hypergeometric function, we have

$$\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\iint u^{a-1}v^{b-1}(1-u-v)^{c-1}$$

$$\left\{\sum_{m=0}^{\infty}\frac{\left(\frac{a+b+c}{2}\right)_m,\left(\frac{a+b+c+1}{2}\right)_m 2^{2m}(xu^2+yuv+zv^2)^m}{(c)_m\ m!}\right\} \text{dudv} \quad (4.1)$$

On using (2.5), (2.6) and (2.7) in the series on the right hand side of (4.1), we pbtain

$${}_2F_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\frac{a+b+c}{2}\right)_{m+n},\left(\frac{a+b+c+1}{2}\right)_{m+n}}{(c)_{m+n}}$$

$$= \frac{2^{2m+2n}u^{2m}x^m(yuv+zv^2)^n}{m!\,n!} \quad (4.2)$$

Again by using (2.5), (2.6) and (2.7) in (4.2) we get

$${}_2F_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\left(\frac{a+b+c}{2}\right)_{m+n+p},\left(\frac{a+b+c+1}{2}\right)_{m+n+p}}{(c)_{m+n+p}}$$

$$\times\frac{2^{2m+2n+2p}u^{2m+n}v^{n+2p}x^m y^n z^p}{m!\,n!\,p!} \quad (4.3)$$

By making use of (4.3) in (4.1) and changing the order of integration and summation, we have

$$\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{2^{2m+2n+2p}\left(\frac{a+b+c}{2}\right)_{m+n+p},}{(c)_{m+n+p}}$$

$$\times\frac{\left(\frac{a+b+c+1}{2}\right)_{m+n+p}x^m y^n z^p}{m!\,n!\,p!}$$

$$\times\left[\iint u^{a+2m+n-1}v^{b+n+2p-1}(1-u-v)^{c-1}\ \text{dudv}\right] \quad (4.4)$$

Which upon applying the results (2.2), (2.8) and (2.9) yields the results (3.1)

The other desired results (3.2) to (3.5) can be proved by the similar method.

V. Conclusion

In this paper some double integral representations have been established for Exton function X_9. A numerous integrals can be obtained for the Extons as well as other hypergepmetric functions. Not only this many new results can be derived which are applicable for Exton's functions.

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